AN EXPLORATION OF THE COURNOT DUOPOLY DYNAMICS

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Abstract

The main aim of this paper is to research the dynamical properties of a Cournot duopoly with relative profits maximizations and costs functions with externalities of the model that was introduced in [A.A. Elsydyna, Appl. Math. Comp 294 (2017) 253-263]. For this purpose the 0-1 test for chaos and approximate entropy were applied to the model supported by the bifurcation diagrams.

Keywords

Cournot Equilibrium, Dynamical System, Chaos, Bifurcation

I. Introduction

In the past decades, plenty of authors were investigated and extensively studied numerous variants of a Cournot oligopoly models from a different point of view. The first treatment of oligopoly was proposed by A. Cournot (1838), for a duopoly, and later on the significant additions to the theory were made exactly one hundred years later by H. von Stackelberg (1938).

It was stated by Theocharis (1960) (see also Palander (1931) page 237) that the oligopoly model produced under constant marginal costs with a linear demand function is neutrally stable for three competitors and unstable for more than three competitors. The argument for this fact can be found in Pu (2007). As discussed in Pu (2007), linear demand functions are very easy to use, but they do not avoid negative supplies and prices, so it is possible to use them only for the study of local behavior. This problem can be solved by using nonlinear demand functions such as piecewise linear functions or other more complex functions, one of which was suggested by Pu (1991) for a duopoly and later by Pu (1996) for a triopoly using iso-elastic demand functions. These types of demand function were later studied by Agiza (1998) and Ahmed et al (1998) for a nonlinear (iso-elastic) demand function and constant marginal costs and it was concluded that this Cournot model for n competitors is neutrally stable if n=4 and is unstable if the number of competitors is greater than five (see also Pu (2007) and Guirao et al (2009)).

Furthermore, in Baiardia et al (2015) an evolutionary model of oligopoly competition where agents can select between different behavioral rules to make decisions on productions. In Bischi et al (2015) a nonlinear discrete-time dynamic model proposed by Farris et al (2005) as a market share attraction model with two firms that decide marketing efforts over time according to best reply strategies with naive expectations. In Brianzonia et al (2015) the authors researched mathematical properties and dynamics of a duopoly with price competition and horizontal product differentiation by introducing quadratic production costs (decreasing returns to scale), thus extending the model with linear costs (constant returns to scale) of Fanti et al (2013). The main aim of Gori, Sodini, Fanti (2015) is a study of local and global dynamics in a nonlinear duopoly with quantity-setting firms and non-cooperative advertising in vestments that affect the degree of (horizon-tally) differentiated products. The paper Gori, Guerrinig, Sodinic (2015) extends the classical repeated duopoly model with quantity-setting firms of Bischi et al (1998) by assuming that production of goods is subject to some gestation lags but exchanges take place continuously in the market.

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Opposing to the previous results attained on homogeneous Cournot models the following are attained on heterogeneous one, such results are not so numerous. The Cavalii et al (2015A) paper studies oligopolies of generic size consisting of heterogeneous firms, which adopt best response adjustment mechanisms with either perfect foresight (rational firms) or static expectations (naive firms). Carraro et al (2015) develops a heterogeneous agents model of set price and inventory with a market maker who considers the excess demand of two groups of agents that employ the same trading rule (i.e. fundamentalists) with different beliefs on the fundamental value. The work Cavalii et al (2016) studies oligopoly models in which firms adopt decision mechanisms based on the best response techniques with different rationality degrees. Heterogeneous Cournot oligopolies of variable sizes and compositions, in which the firms have different degrees of rationality, being either rational firms with perfect foresight or naive best response firms with static expectations are explored in Cavalii et al (2015B). The authors in Radi et al (2015) consider a Schelling-type segregation model with two groups of agents that differ in some aspects, such as religion, political affiliation or color of skin.

In Lampart (2017) it is considered the variety of the Cournot oligopoly models, from classical homogeneous to heterogeneous, and the Cournot points are constructed. Their stability is discussed while the number of players is increasing.

The main aim of this paper is to research the dynamical properties of the Cournot duopoly model introduced by Elsadany (2017). For this purpose the 0-1 test for chaos and approximate entropy is applied to the model in dependence of the state parameters.

II. Cournot duopoly model

In this paper, a Cournot duopoly model introduced by Elsadany (2017) is researched. This model consists of two players trying to maximize their profit p_i through adjustment of the supplied product quantity q_i.

In this case each company produce differentiated product varieties with differentiation given by the parameter b. If the b is equal to 1, the products are perfect substitutes and the products are completely differentiated for the b equal 0.

The resulting game is given by the two equations describing the adjustment of the supplied quantity for each firm

$$\begin{aligned} q_1(t+1) &= q_1(t) + \alpha_1 q_1(t)[a - c_1 - 2q_1(t) - (b_1 - b_2)q_2(t)], \\ q_2(t+1) &= q_2(t) + \alpha_2 q_2(t)[a - c_2 - 2q_2(t) - (b_2 - b_1)q_1(t)] \end{aligned} \quad (1)$$

where $q_i(t)$ is supplied quantity of product variety i produced by the player i at the time t, α_i is the adjustment parameter of the player i, c_i is the marginal cost of the player i.

The deceptive derivation of this model was done Elsadany (2017).

It is easy to get the following equilibriums E_i of the model (1), see Elsadany (2017):

$$E_1 = (0,0), \qquad E_2 = \left(\frac{a - c_1}{2}, 0\right), \qquad E_3 = \left(0, \frac{a - c_2}{2}\right),$$

$$E_4 = \left(\frac{(2 - b_1 + b_2)a - 2c_1 + (b_1 - b_2)c_2}{4 + (b_1 - b_2)^2}, \frac{(2 - b_2 + b_1)a - 2c_2 + (b_2 - b_1)c_1}{4 + (b_1 - b_2)^2}\right).$$

The point E_1, the trivial equilibrium, is a repelling node; the boundary equilibriums E_2 and E_3 are both saddle points. Finally, the Cournot-Nash equilibrium E_4 is possibly sink, source, saddle, or non-hyperbolic in dependence of the system (1) parameters while crossing the Flip or Neimark-Sacker bifurcation border, see Elsadany (2017) for more details and Devaney (2003) for the introduction to the theory of dynamical systems to get standard definitions and properties used here.

The stability means that the value of (α_1, α_2) is in the stable region, output of two firms q_1, q_2 will achieve the Cournot--Nash equilibrium quantity after a number of games. Nevertheless, once the increase of $\alpha_1$or $\alpha_2$moves the point (α_1, α_2) out of the stability region, more complex phenomena of outputs evolution will occur such as bifurcation and chaos that will be the main aim of our investigation.

III. Methods of analysis

This section is devoted to the introduction of two dynamical properties detection methods, tools that will be used in the sequel - the 0-1 test for chaos and approximate entropy.

The 0-1 test for chaos

The 0-1 test for chaos was proposed by Gottwald and Melbourne (2004) to discern time series with regular and chaotic dynamics. The main advantage of the test is, that it returns the value between 0 and 1, where 0 represent regular dynamics and 1 chaotic dynamics.

First the translation variables are computed for at least 100 different $c \in (\pi/5, 4\pi/5)$:

$$p_c(n) = \sum_{j=1}^{N} x(j)\cos(jc), \tag{2}$$

$$q(n) = \sum_{j=1}^{N} x(j)\sin(jc), \tag{3}$$

where *x(j)* is the *j*-th element of time series *x*, $j \in (1,2,\dots,N)$.

To estimate boundedness of the object given by the translation variables p_c and q_c, the mean square displacement is computed and defined as follows:

$$M_c(n) = \lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} |p_c(j+n) - p_c(j)|^2 + |q(j+n) - q(j)|^2. \tag{4}$$

In practice *n* must be much smaller than *N* for the limit to converge. Gottwald and Melbourne (2009) claim, that *n = N/10* is enough.

The bounded M_c indicates that the object in *p-q* plane is bounded and therefore the dynamics of the series is regular.

Otherwise, the M_c is increasing and the dynamics of the time series is chaotic.

To quantify whether the M_c is bounded or increasing the growth rate K_c is determined as a correlation of M_c and steadily increasing series:

$$K_c = \mathrm{corr}(t, m) = \frac{\mathrm{cov}(t,m)}{\sqrt{\mathrm{var}(t)\mathrm{var}(m)}} \in [-1,1], \tag{5}$$

where $t = (1,2,\dots,n)$ and $m = (M_c(1), M_c(2),\dots, M_c(n))$.

If the K_c is close to *0* the M_c is bounded and therefore the dynamics of the time series is regular. If the K_c is close to *1* the time series has chaotic dynamics.

As the final value of the test, the median of multiple K_c is taken, since for some special values of *c* there may be false results of the test.

Approximation entropy

Approximation entropy is a pretty robust measure of complexity or irregularity. Advantages of approximation entropy are robust estimations with short time series and detection of differences in complexity in the same system with different parameter, see e.g. Pincus et al (1991), Sabeti et al (2009). To compute the Approximation entropy it is necessary to define two parameters: embedding dimension *m* and neighbourhood threshold *r*.

Let $x(t) \in \mathbb{R}$ for $t = \{1,2,\dots,n\}$ be a time series with *n* observations. Then embedded vector *X(t)* at time *t*, is defined as $X(t) = [x(t), x(t+1), x(t+2),\dots, x(t+(m-1))]$, where *t* is the observed time and *m* is embedding dimension.

Maximum distance of embedded vectors is computed as follows:

$$D(i,j) = d\big(X(i), X(j)\big) = \max_{k=0,1,2,\dots,m-1} |x(i+k) - x(j+k)|, \tag{6}$$

for $i,j = \{1,2,\dots,N-(m-1)\}$.

Compute the thresholded version of the distance with threshold given by r:

$$d_r(i,j) = \begin{cases} 1, D(i,j) < r, \\ 0, \text{otherwise}, \end{cases} \tag{7}$$

for $i,j \in \{1,2,\dots,N-(m-1)\}$.

Compute $C_i^m(r)$ as a ratio between points close to the i and the number of vectors.

$$C_i^m(r) = \frac{\sum_{j=1}^{N-(m-1)} d_r(i,j)}{N-(m-1)}. \tag{8}$$

Then compute the average of logarithm of all the $C_i^m(r)$

$$\phi^m(r) = \frac{1}{N-(m-1)} \sum_{i=1}^{N-(m-1)} \ln C_i^m(r). \tag{9}$$

Finally, Approximation entropy for the finite time series with N data points is computed as

$$ApEn(m,r,n) = \phi^m(r) - \phi^{m+1}(r). \tag{10}$$

Pincus (1991) suggest to use at least 10^3 observations for robust estimation.

IV. Main results

This section is focused on the exploration of the dynamical characteristics of the model given by Equation (1). These characteristics are considered in terms of changing the adjustment rate of the player α_1.

The experiments are done for the parameters a and c_i fixed to the following values $a=6$, $c_1=0.1$ and $c_2=0.2$. Two types of the outputs will be provided, the bifurcation diagrams for the given settings of α_2 and b_i and the results of the 0-1 test for chaos and approximation entropy computed for the time series of q_1, q_2 and series *comb* given by

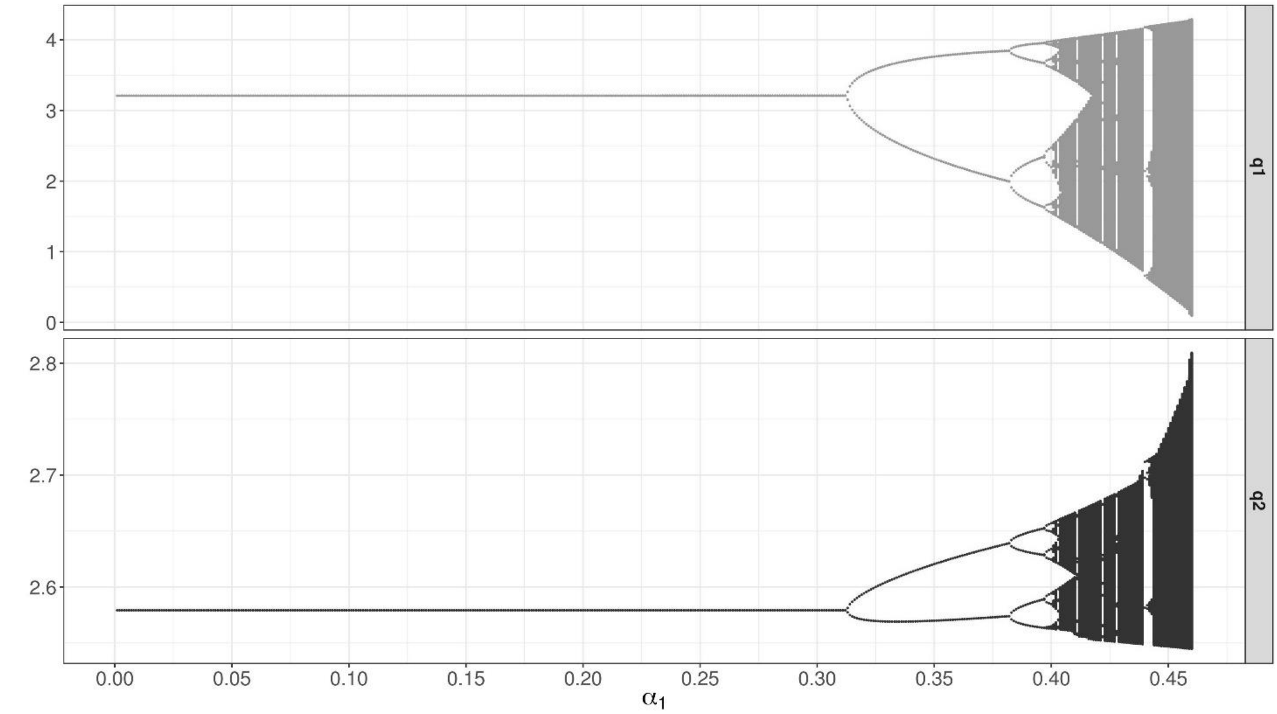
$$comb(j) = \sqrt{q_1^2(j) + q_2^2(j)}.$$

Bifurcation diagrams are computed for the 10^5 iterations and the last 2×10^4 points are plotted for each value of the α_1. For the 0-1 test for chaos and approximation entropy a 1.2×10^5 iterations were computed, and the last 10^5 iterations were used for the computation of the tests. In all the tests the step of consecutive values of $\alpha_1$is 0.0005.

For the first test, experiment parameters are set to the following: $\alpha_2=0.1$, $b_1=0.1$ and $b_2=0.3$. Parameter $\alpha_1$belongs to the interval $(0, 0.46)$. Initial points for the computation were $q_1 = 2 + \sqrt{2}/2$ and $q_2 = 2 + \sqrt{3}/3$.

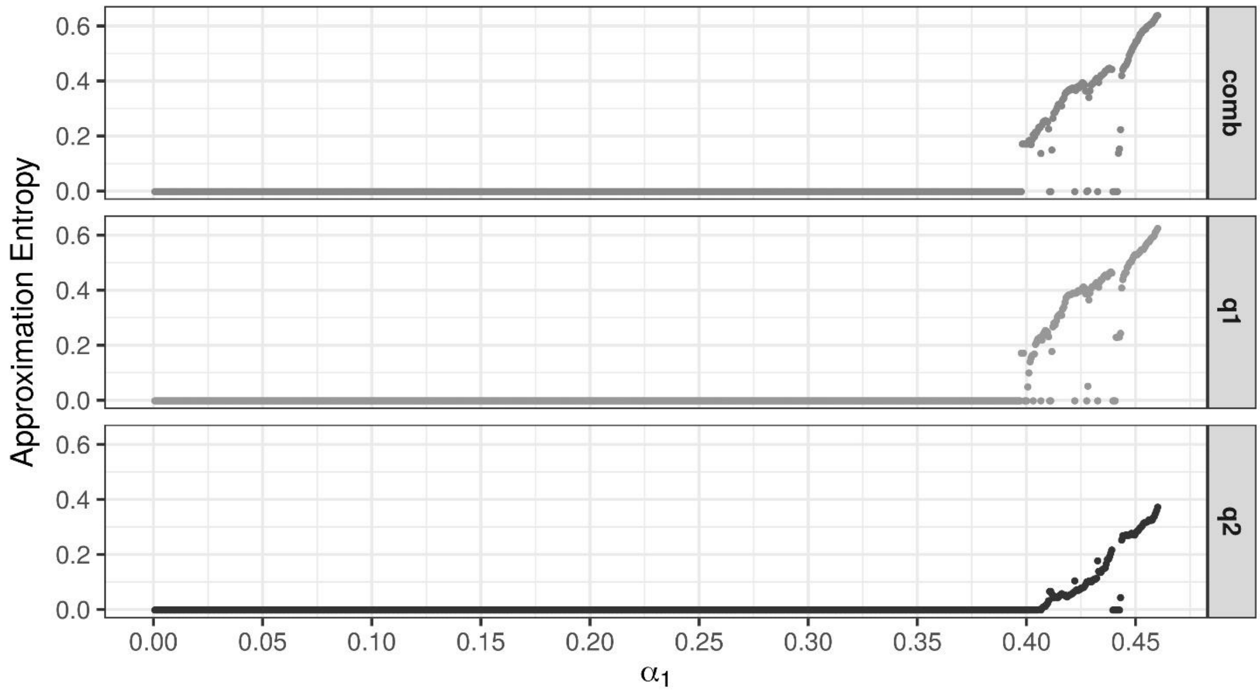
In Figure 1, there are the bifurcation diagrams for the variable q_1 and q_2. The bifurcation diagrams are shown on different y-scales, so the changes in dynamics are clear for both players. It may be seen that for the small values of α_1 both players will get to the stable state, where the quantities of the product are fixed. First bifurcation occurs at $\alpha_1 \approx 0.31$ and then the period doubling bifurcations occur. At approximately $\alpha_1 > 0.4$ there is chaotic behavior present. This is further supported by the Figure 2 and Figure 3. It is shown, that when the $\alpha_1 \approx 4$ the approximation entropy starts increasing rapidly and the 0-1 test for chaos flip from 0 to 1 for all the cases. In Figure 4, it is shown the attractor for the $\alpha_1=0.41$ (left) and $\alpha_1=0.46$ (right). Difference between the two attractors shows how the attractor expands with the increasing value of α_1.

Figure 1 **Bifurcation diagram dependent on the α_1 for q_1 [top] and q_2 [bottom] the with the settings α_2=0.1, b_1 =0.1, and b_2=0.3.**



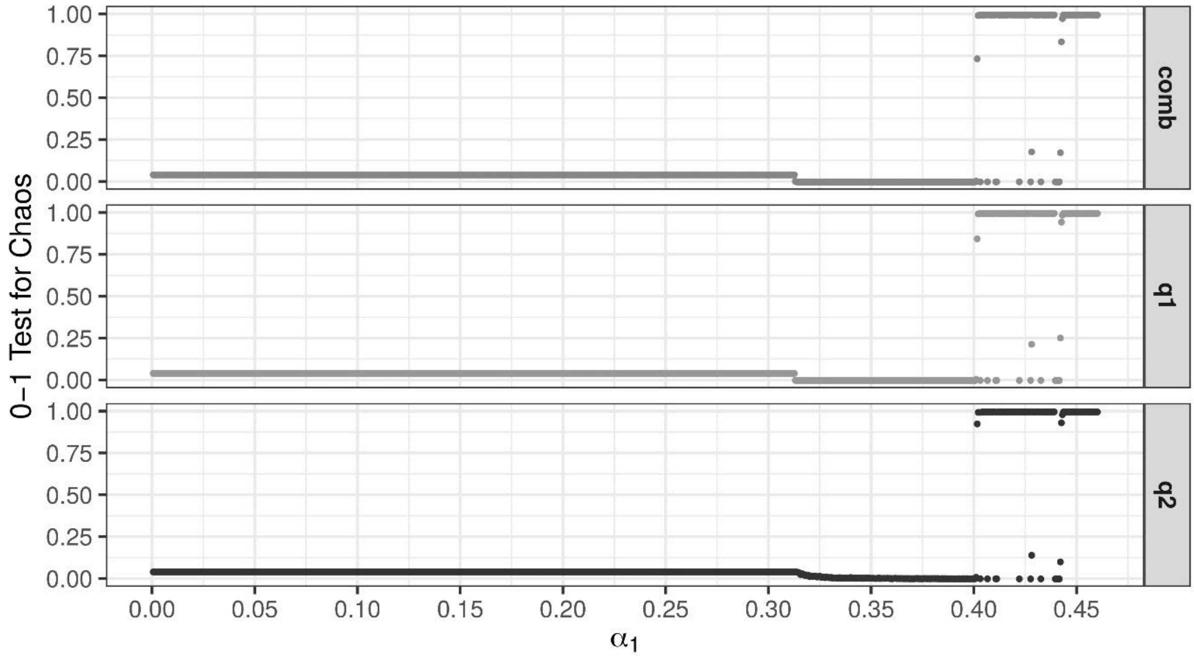
Source: Own processing

Figure 2 **Approximation entropy dependent on the α_1 with the settings α_2=0.1, b_1=0.1, and b_2=0.3.**



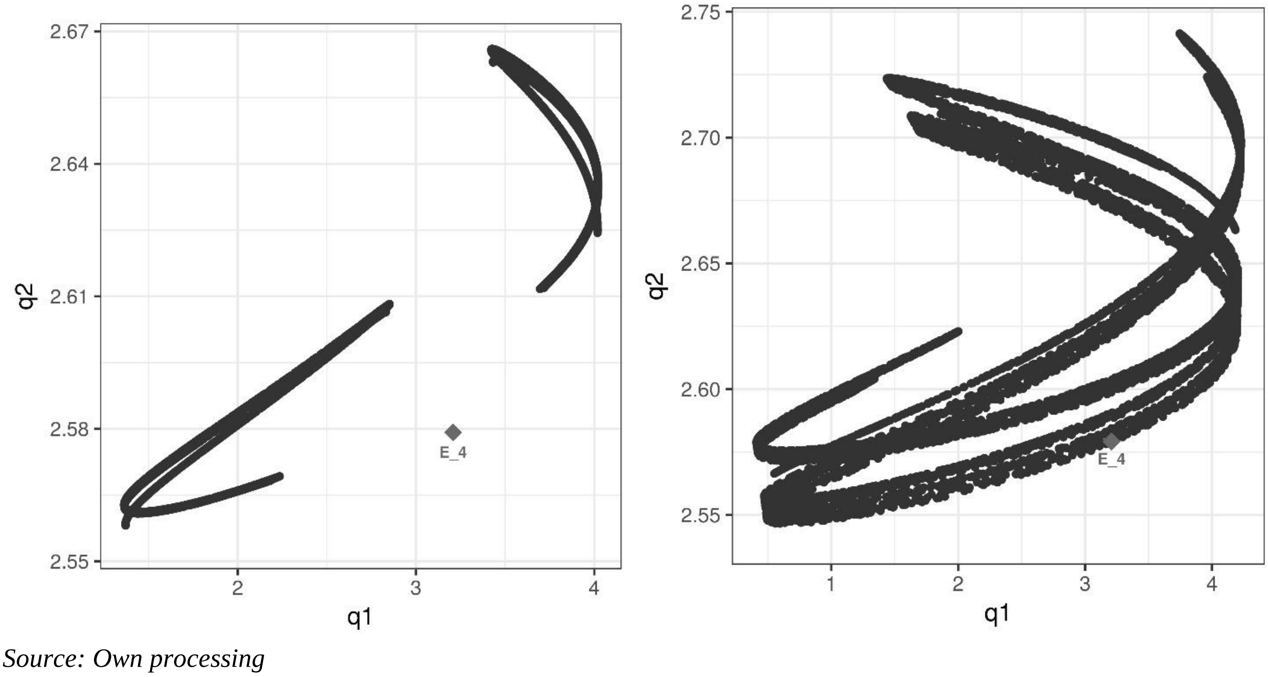
Source: Own processing

Figure 3 **The 0-1 test for chaos dependent on the α_1 with the settings α_2=0.1, b_1=0.1 and b_2=0.3.**



Source: Own processing

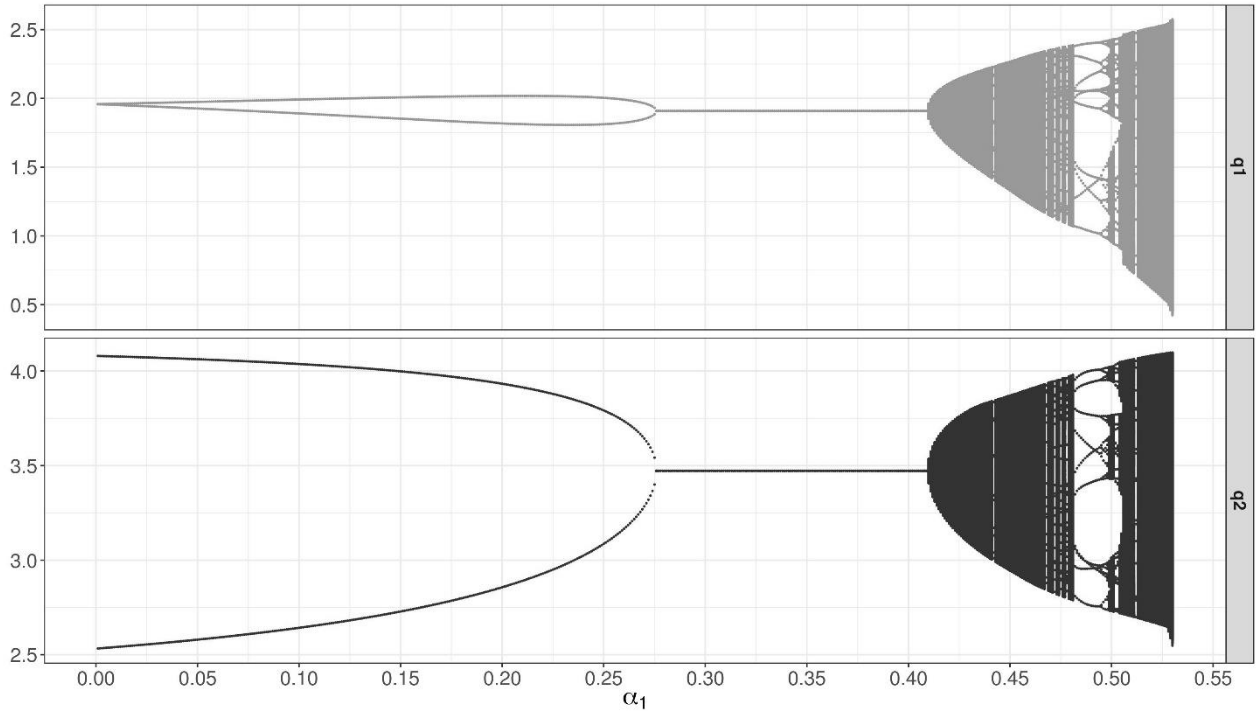
Figure 4 **Attractor in the q_1-q_2 plane, for the iterations of the model (1) with the settings α_2=0.1, b_1=0.1, and b_2=0.3, and α_1=0.41$ (left) and α_2=0.45 (right).**



Source: Own processing

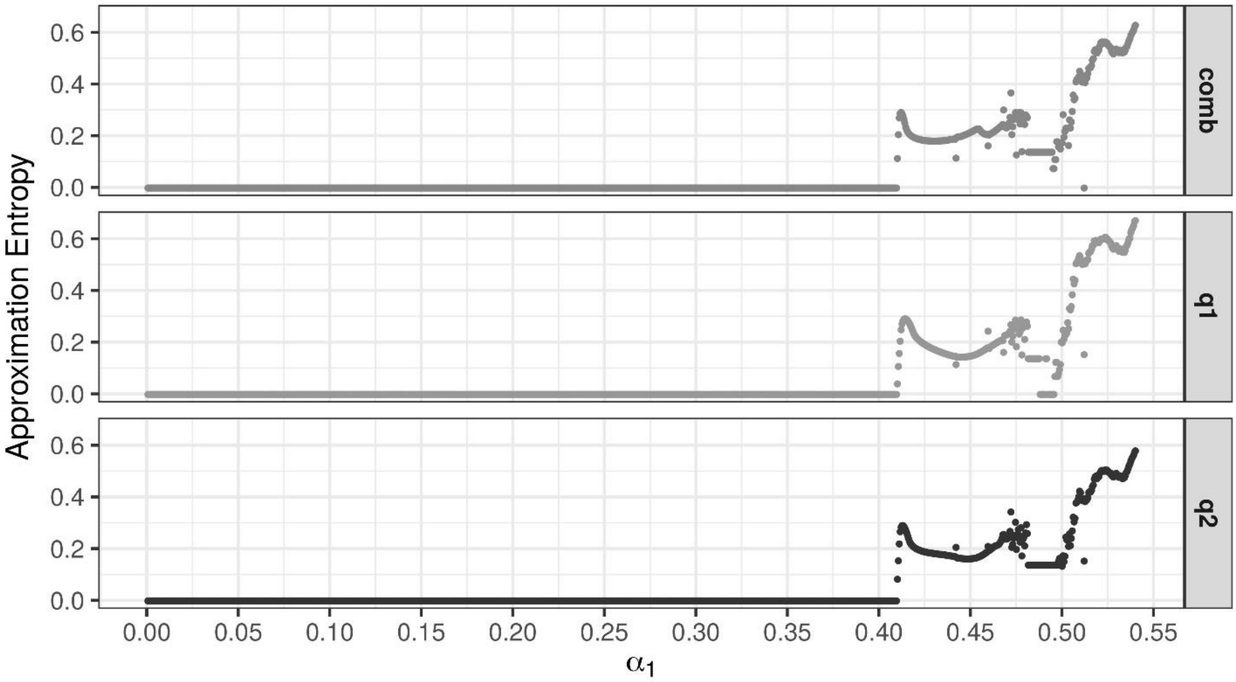
Second parameter set tested in this paper is α_2=0.32, b_1=1 and b_2=0.4. Initial points for the computation were $q_1 = 2 + \sqrt{2}/2$ and $q_2 = 2 + \sqrt{3}/3$. The bifurcation diagrams for this system can be found in Figure 5. In this case there is a periodic behavior for the low values of α_1, followed by a single value stable state. At $\alpha_1 \approx 0.41$ the dynamics of the system start to be much more complicated. At this point the approximation entropy start to increase (see Figure 6), but the 0-1 test for chaos still provides results close to 0 (see Figure 7). The results of the 0-1 test for chaos are suggesting regular dynamics in case of the more complicated system. To more inspect what is happening at this point a visualization of the attractor is necessary. In the following figures the plots of q_1 to q_2 are shown for the last 10^4 iterations of the 10^5 iterations computed. In Figure 7 it is shown the attractor for the α_1 = 0.41 (left) and α_1= 0.51 (right). In the first case, the attractor is created by the single repeating path in the q_1-q_2 plane. This is probably the reason why the 0-1 test for chaos determined, that the system is regular. In the second case the dynamics of this system are much more complex and such dynamics are recognized as chaotic even by the 0-1 test for chaos.

Figure 5 Bifurcation diagram dependent on the α_1 for q_1 [top] and q_2 [bottom] the with the settings α_2=0.32, b_1 =1, and b_2=0.4.



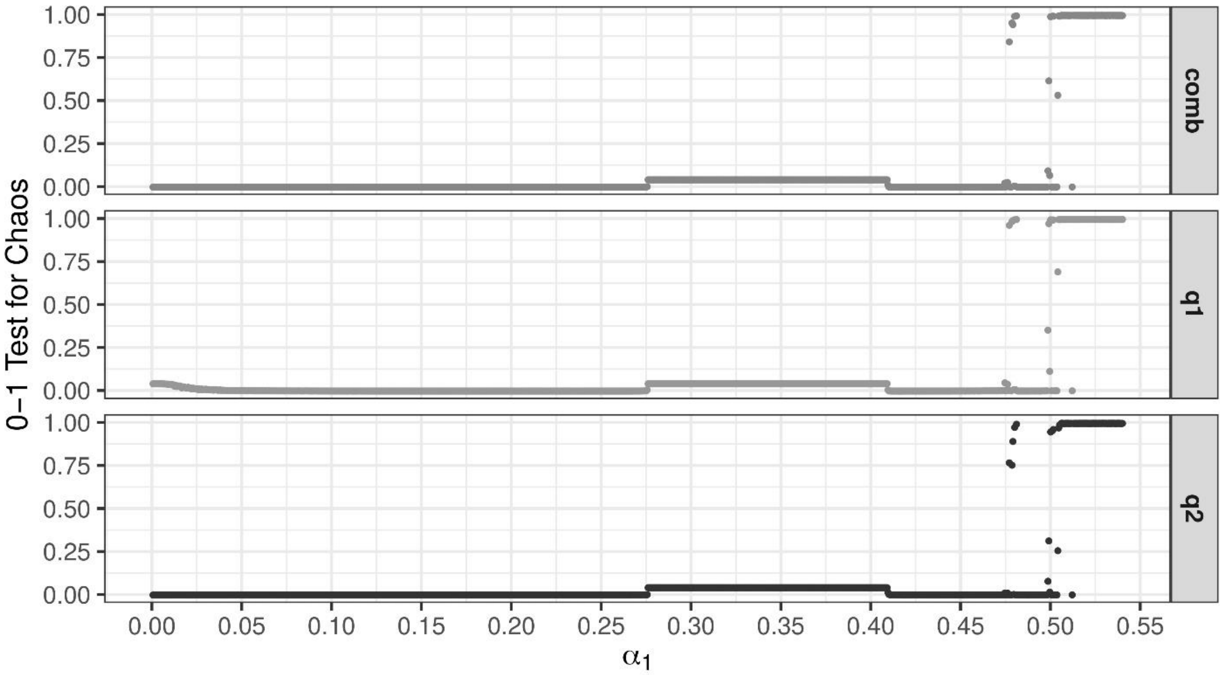
Source: Own processing

Figure 6 Approximation entropy dependent on the α_1 with the settings α_2=0.32, b_1=1, and b_2=0.4.



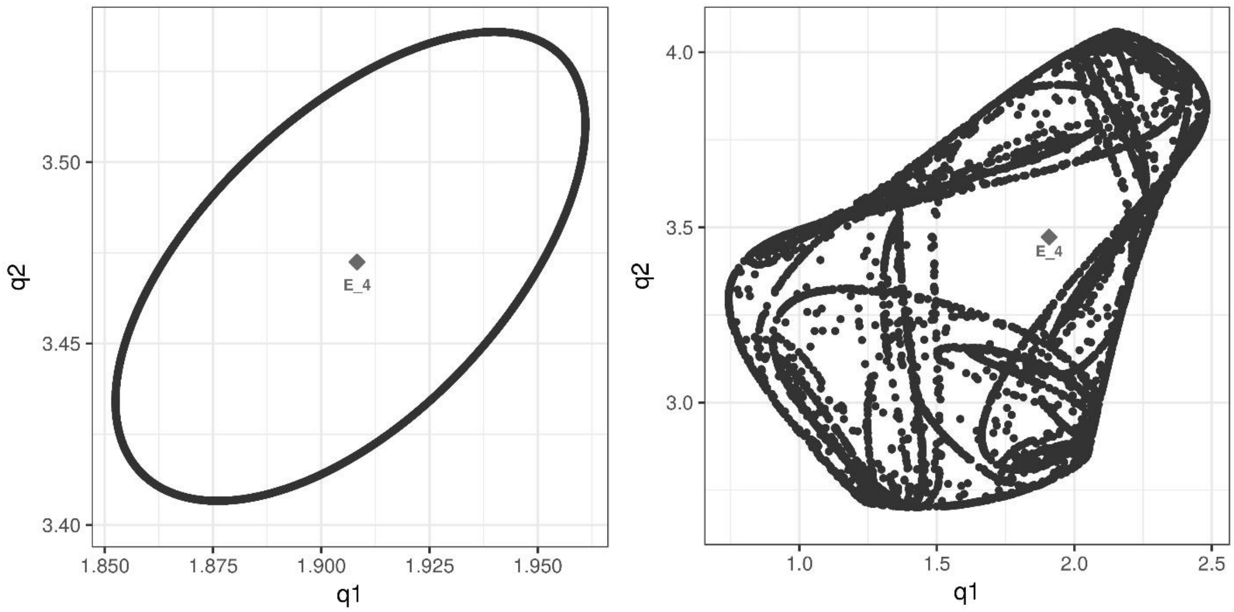
Source: Own processing

Figure 7 The 0-1 test for chaos dependent on the α_1 with the settings α_2=0.32, b_1=1 and b_2=0.4.



Source: Own processing

Figure 8 Attractor in the q_1-q_2 plane, for the iterations of the model (1) with the settings α_2=0.32, b_1=1, and b_2=0.4, and α_1=0.41$ (left) and α_2=0.51 (right).



Source: Own processing

IV. Conclusion

This paper focuses on the dynamical properties of the Cournot duopoly game with relative profits maximizations and costs function with externalities. The dynamics was studied using the 0-1 test for chaos, bifurcation diagrams, and approximation entropy.

It is shown in Sec. Main results that the model shows regular (periodic) behavior as well as irregular (chaotic) for a suitable choice of the model parameters.

Acknowledgements

This work was supported by The Ministry of Education, Youth and Sports from the National Programme of Sustainability (NPU II) project ``IT4Innovations excellence in science - LQ1602" and by the IT4Innovations infrastructure which is supported from the Large Infrastructures for Research, Experimental Development and Innovations project ``IT4Innovations National Supercomputing Center – LM2015070" and by the SGC grant No. SP2018/173 "Dynamics systems problems and their implementation on HPC ", VŠB - Technical University of Ostrava, Czech Republic, and by Grant of SGS No. SP2018/165, VŠB - Technical University of Ostrava, Czech Republic.

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