COURNOT OLIGOPOLY AND CHAOS THEORY – IMPACT FOR ECONOMIC MODELLING

Veronika Nálepová[1], Judita Buchlovská Nagyová[2], Marek Lampart[3]

Abstract

The main aim of this paper is to show the importance of chaos theory in economics by using the example of the Cournot oligopoly model. The purpose is not to describe the essence of chaos theory as such, which has been the subject of many mathematical or physical studies, but to point out the insights and significance of chaos theory used in the framework of the Cournot oligopoly. This paper is intended to be a reflection on the relevance of chaos theory in oligopoly games, especially in an economic context. The paper briefly describes the historical development of the use of chaos theory in the Cournot oligopoly model, its basic mathematical definition, and a simple example of an application of chaos theory. Understanding the subsets of chaos theory in the context of economics highlights the importance of a deterministic approach to economics, the impossibility of making long-run predictions, but the ability to correct and manage the system to avoid complex oscillations. Oligopolistic structures, with their simpler system than perfect congruence, allow chaos theory to be implemented appropriately.

Keywords

Cournot Oligopoly, Chaos Theory, Economic Modelling, Nonlinear Dynamic System, Determinism

I. Introduction

Chaotic behaviour in economics was pointed out as early as 1988 by J. Kelsey, who states that „*Economics contains the following paradox. In microeconomics all economic variables are seen as produced by the rational decisions of maximizing agents. Hence in microeconomics variables are completely deterministic. In macroeconomics, however, economic variables are frequently viewed as being random. How can the same variables be random and deterministic at the same time? This paradox may be explained if the economic system is chaotic.* "

The new theories develop as a causal consequence of the need to improve older ones. New paradigms were also brought about by chaos theory. The basic premise of chaos theory is determinism and nonlinearity. The question, of course, is whether processes in the real world can be considered deterministic or stochastic. The answer, of course, is not clear-cut, and both approaches have their place in scientific history and the present. The interplay of events that occur in the system of our society from the perspective of chaos theory is hardly ever random. Even more so from the perspective of economics, all processes are rather dynamic in nature and capturing the development of the economy by means of singular static measures is somewhat vague. Yet it is change and its consequences that interest us economists most. The ability to manage, predict, anticipate, or structure change is what matters in economic processes, or in strategic decision-making processes.

In oligopolistic corporate structures, each firm is aware that its profits depend on the scale of production of its competitors. Strategic management is therefore needed. Oligopolies are very complex market structures where there is no single descriptive model for different types of behaviour. According to Vega-Redondo (1997), it has been shown that the Curont-Nash equilibrium is not

[1] PRIGO University, V. Nezvala 801/1, 73601 Havířov, Czech Republic. E-mail: veronika.nalepova@prigo.cz.
[2] IT4Innovations, VSB - Technical University of Ostrava, 17. listopadu 2172/15, 708 00 Ostrava, Czech Republic; Department of Applied Mathematics, VSB - Technical University of Ostrava, 17. listopadu 2172/15, 708 00 Ostrava, Czech Republic. E-mail: judita.buchlovska.nagyova@vsb.cz.
[3] IT4Innovations, VSB - Technical University of Ostrava, 17. listopadu 2172/15, 708 00 Ostrava, Czech Republic; Department of Applied Mathematics, VSB - Technical University of Ostrava, 17. listopadu 2172/15, 708 00 Ostrava, Czech Republic. E-mail: marek.lampart@vsb.cz.

evolutionarily stable. This realization is very important in the context of the use of chaos theory in strategic management and the use of evolutionary game theory.

The confirmation of chaos in economics has brought a new perspective on how to think about an economic system. Chaos theory recognizes that the future is unknown, but it allows for the possibility of realizing a range of future states. Furthermore, it suggests that complete and accurate information, so necessary for rational decision making, is unattainable and that the past is not an accurate guide to the future. However, most studies that investigate the dynamics of the duopoly game focus on describing the mathematical system and the importance in economic sense is often neglected. The aim of the present contribution is to highlight the importance of chaos theory in economics by using the Cournot oligopoly model as an example.

II. Basic Knowledge - Chaos theory, Cournot Oligopoly and Modelling

Cournot introduced the theory of oligopoly in 1838, where each oligopolist assumes that other firms hold their output constant and all oligopolists choose quantities based on the output of others in order to maximize profits. However, the oligopoly game becomes complex when players adopt a dynamic strategy and choose different expectation rules to adjust their output. There are three different expectations, namely naive, bounded rational and adaptive. Based on the expectations, the opponent's strategy then chooses the optimal response to the size of the output. We also consider whether the players' strategy will be heterogeneous (different strategies for each player) or homogeneous (both players choose the same strategy) or whether the strategy will evolve over time. Another aspect is the choice of different demand or cost functions. These assumptions are then used to derive modelling approaches, which are further extended by game theory or evolutionary game theory.

Loernz (1963) discovered in a weather simulation that a small change in initial conditions will cause a large change in future predictions. He thus confirmed the conjectures and work of his predecessors (Maxwell, 1876 or Poincaré, 1913). Until then, scientists believed that a small deviation would cause only a small change in the prediction. This revolutionary discovery was fundamental for chaos theory and changed the view of the possibilities of deterministic predictability. Deterministic chaos, then, is behaviour that looks random but is deterministic in nature. Chaos is then seen as a typical mathematical property of a dynamical system. Chaos thus appears as unstable aperiodic behaviour in nonlinear dynamical systems (Kellert, 1993).[1]

The Cournot oligopoly is represented by discrete dynamical systems. A dynamical system is composed of a phase (state) space whosě coordinates describe the state of the system at a given time. In the case of Cournot oligopoly, it is the equilibrium quantity of output of the observed firms, or Cournot-Nash equilibrium. Furthermore, from the dynamic terms that describe the change of this system over time, it is the change of the system based on the control parameters, which can be the speed of adjustment of a given strategy. Thus, the dynamics is embodied by the gradual adjustment among competitors. The state of the system is then described by a vector that lies entirely in the state space. If the system is defined as a sequence of moves, then the outcome (orbit) may be a fixed point, equilibrium, periodic oscillation, quasiperiodic oscillation, or chaos with no identifiable pattern.

Puu (1991) was one of the first to detect chaotic behaviour in the duopoly game, which was predicted by Rand (1987). Here he assumed isolestic demand and constant unit costs of production. The conclusion of his paper was that the model can generate persistent motion, periodic or chaotic. In particular, if appropriate nonlinearities are included the model is capable of producing chaotic dynamics. His work has been followed by many other authors (Kopel, 1998; Agiza and Elsadany, 2003; Bischi et al., 2000; Bischi and Naimzada, 2000), where a mix of conditions[2] provides very interesting results on how dynamic processes work in oligopoly models and how changing individual

[1] The definition of deterministic chaotic behaviour is not unambiguous for a more detailed discussion on this topic see viz Bishop (2017).

[2] The level of rationality of the players (naive, adaptive, boundedly rational), the demand function (linear or isoelastic), the cost function (linear or quadratic) or the number of players (duopoly, triopoly or quadropoly).

parameters changes the structure of the system. Other authors focus on chaos control in duopoly games (see Du et al., 2010; Agiza et al. 2003; Chen and Chen, 2017; Lampart and Lampart; 2020). A comprehensive overview of the development of dynamic oligopoly theory is then provided by Rosser (2002).

We could list other publications that investigate system dynamics or chaotic behaviour of duopoly or focus on chaos management in duopoly games. However, this is not essential for the purposes of this paper. Obviously, this discipline relies primarily on good mathematical knowledge, but we must not forget that this is, after all, an economic field and here the contribution should be substantial. This fact is not considered in most articles. The monographs by Puu (2011) or Puu et al. (2002) come closest to a comprehensive account. However, understanding chaos theory entails the need for at least a brief basic definition of the mathematical background.

III. Fundamentals of chaos mathematical structure

In this section, we present the tools, which can be used for qualifying and quantifying the dynamical properties of the investigated model. First of all, we recall the fundamental concept of a dynamical system. A *dynamical system* is standardly defined as a pair (X, f) where X is the state space (usually a compact metric space) and $f\colon X \to X$ is a continuous map. The dynamics is then given by iterations of f, that is $f^0 = id_X$ and

$$f^n = \underbrace{f \circ f \circ \dots f \circ}_{n-times} f$$

is n-fold composition of f. So, to a given starting point, denoted x_0 and called an $initial\ point$, one can construct a sequence:

$$x_0, f^1(x_0), f^2(x_0), \dots$$

that is called a *trajectory*. One can then observe that if $f(p) = p$ than $f^n(p) = p$ for any $n \in N$. Such point p is called a *fixed point* (or equilibrium point), and if for $y \in X$ $f^n(y) = y$ and $f^k(y) \neq y$ for any $1 < k < n - 1$ the point is called and n-periodic point. The stability of such points can be easily observed when f is differentiable using test on absolute values of the eigenvalues (Devaney, 1989).

Trajectories that are generated by a periodic point correspond to a regular behaviour. Nevertheless, much more complex cases can appear. The task is to detect them and to measure their complexity - this is a task of the theory of dynamical systems on which researchers focus for more than a century.

For the purpose of this paper, we use sensitivity on initial conditions, bifurcation analysis, the maximal Lyapunov exponent and the 0-1 test for chaos. The use of the methods is shown on a generic example of the well-known logistic map (Devaney, 1989 or May, 1976), given by the equation (1)

$$x_{n+1} = \mu x_n(1 - x_n),$$

where $x \in I = [0, 1]$ and $\mu \in [0, 4]$.

3.1 Sensitive dependence on initial conditions

Sensitive dependence on initial conditions describes a phenomenon in which a small difference in the initial state of the system can overtime lead to considerably different results. A map f on an interval I has sensitive dependence on initial conditions if there exists $\delta > 0$ such that for any $x \in I$ there exists y from the neighborhood of x such that $|f^n(x) - f^n(y)| > \delta$. The simplest example can be $f(x) = cx$.

Figure 1 demonstrates the sensitivity on initial conditions on the example of the logistic map (1) with parameter $\mu = 4$ and two different initial conditions, $x_{01} = \sqrt{2}/2$ and $x_{02} = \sqrt{2}/2 + 10^{-9}$, red and blue respectively. We observe that by slight change of the initial condition the trajectories of the system vary significantly after 25 iterations. The distance $d(f_\mu^n(x_{01}), f_\mu^n(x_{02}))$ between these trajectories is reaching $diam(X)$, in this case 1, see Figure 2, where $f_\mu(x)$ is the function of the system (1) and d stands for the Euclidian metric in $\mathbb{R}$.

Figure 1 Two trajectories of the logistic map (1) obtained by iterating the system from two different initial conditions x_{01} and x_{02}.

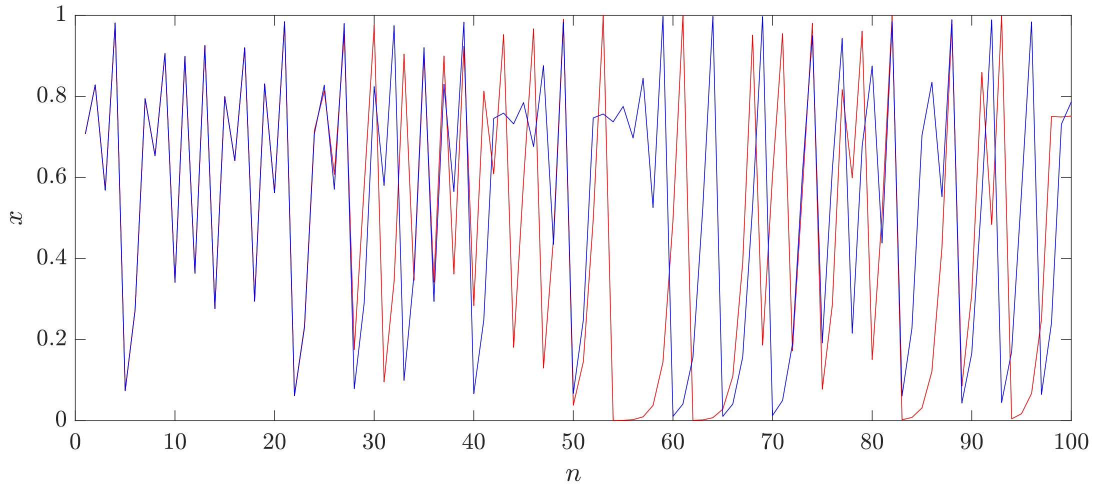
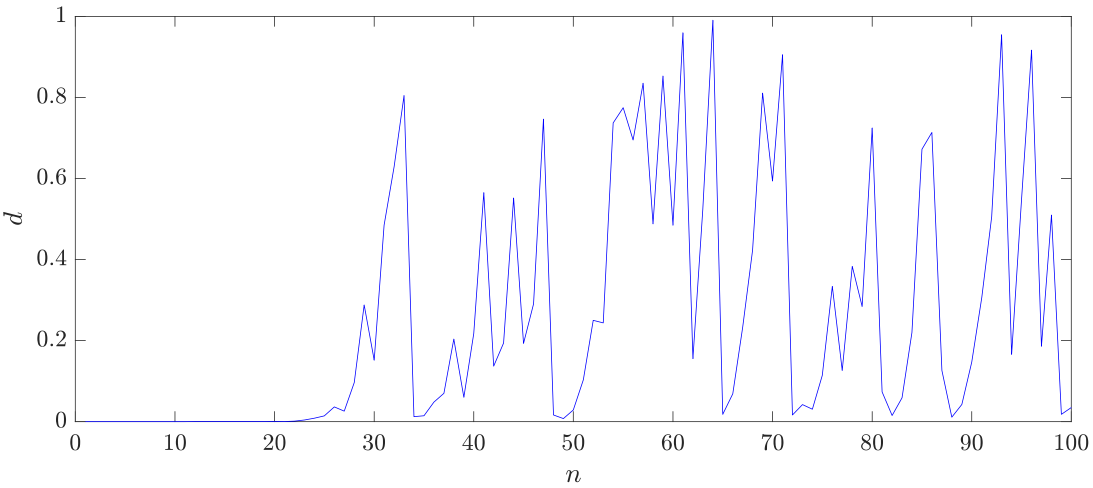


Figure 2 The difference of trajectories of the logistic map (1) obtained by iterating the system from two different initial conditions x_{01} and x_{02}.



3.2 Bifurcation diagram

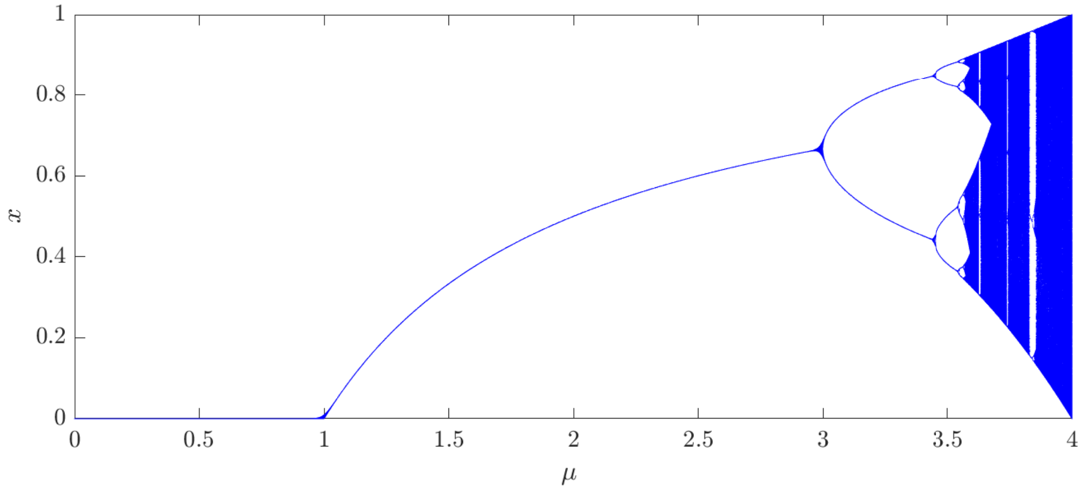
Bifurcations are sudden changes in the geometry or topology of the system's attractor at a critical value of a parameter. The most common type is the period doubling bifurcation. At a specific critical value of the parameter, a stable orbit of period p becomes unstable, and an orbit of period $2p$ appears. This can happen multiple times with more period doublings occuring with increasing values of the parameter, until infinite period appears, and the dynamics become chaotic. The values of the parameter where period doublings occur are important. The point μ_k, where the k-th period doubling occurs scale with the distance from the two points where the period doubling occured. The values of ratio:

$$\delta_k = \frac{\mu_{k-1} - \mu_{k-2}}{\mu_k - \mu_{k-1}}$$

for the k-th period doubling converge towards the constant $\delta \approx 4.669$, called the Feigenbaum constant (Feigenbaum, 1978).

The bifurcation diagram, see Figure 3, is the representation of changes that a dynamical system (1) undergo depending on a change in its parameter. The diagram is constructed as the values visited or approached as a function of the bifurcation parameter μ.

Figure 3 The bifurcation diagram of the logistic map (1)



3.3 The Lyapunov exponents

One of the most renowned methods for determining the dynamical properties of the system is by computing Lyapunov exponents (Rosenstein et al., 1993; Wolf et al., 1985). As we know from the concept of the sensitivity on the initial conditions, trajectories which were initially close to each other, may diverge over time. If this divergence is exponentially fast, we may speak of ”chaos”. More precisely, if the Lyapunov exponent is negative, the trajectory is regular (periodic). If it is zero, bifurcation occurs, and finally if it is positive, ”chaos” appears. Chaos is then defined as the case, when the Lyapunov exponent is positive. We can compute as many Lyapunov exponents for a dynamical system as there are phase space dimensions. We will work with the largest one, the maximal Lyapunov exponent.

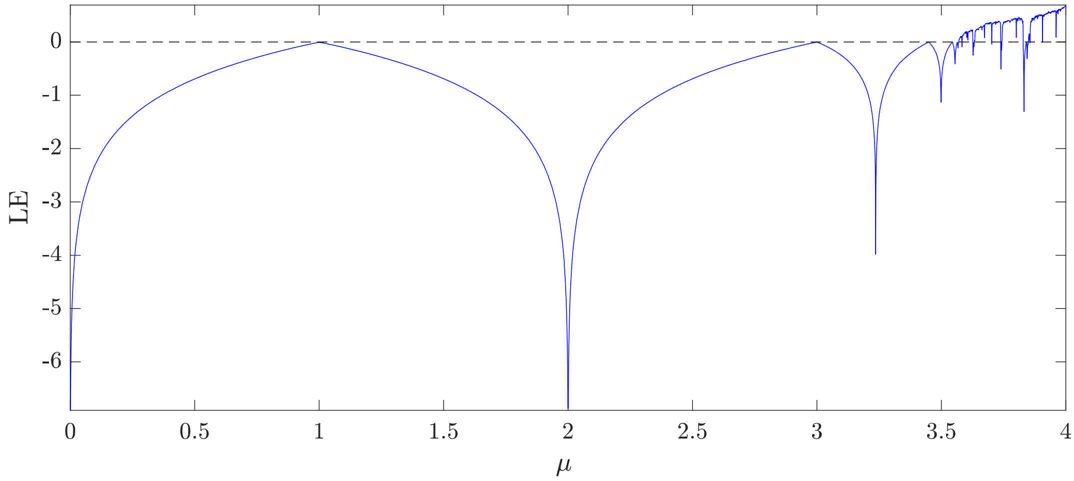
The Lyapunov exponent for a one-dimensional discrete dynamical system is computed as:

$$\lambda = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln[f'(x_i)],$$

under the assumption that this limit exists, and f is differentiable.

Figure 4 represents the changes of the maximal Lyapunov exponent (MLE) for the logistic map (1) depending on different values of the parameter μ.

Figure 4 Values of the Lyapunov exponent (LE) versus the parameter μ of the logistic map (1).



3.4 The 0-1 test for chaos

The 0-1 test for chaos, introduced by Gottwald and Melbourne (2009) is one of the methods for distinguishing between regular and chaotic dynamics of a deterministic system. This test works directly with experimental data, or time series generated by systems of ordinary or partial differential equations. The test returns values close to either 0 or 1, with 0 corresponding to regular dynamics and 1 to chaotic dynamics. With its easy implementation, evaluation, and wide range of application, using this tool for detecting chaos is becoming more popular in various scientific fields (Buchlovská Nagyová et al., 2020; Lampart and Zapoměl, 2019; Halfar, 2021).

The algorithm for the 0-1 test for chaos (Gottwald and Melbourne, 2009) is the following. Given the observation $\phi(j)$ for $j = 1, 2, \dots, N$ and a suitable choice of $c \in (0, 2\pi)$, the following translation variables are computed:

$$p_c(n) = \sum_{j=1}^{n} \phi\,(j)\cos(jc),$$

$$q_c(n) = \sum_{j=1}^{n} \phi\,(j)\sin(jc),$$

for $n = 1, 2, \dots, N$.

The variables p_c and $q_c$are bounded if the movement is regular and unbounded, like a Brownian motion, for chaotic dynamics. The idea for the 0-1 test, first described in Gottwald and Melbourne, (2009), is that the boundedness or unboundedness of the trajectory $\{(p_j, q_j)j \in [1, N]\}$ can be studied through the asymptotic growth rate of its time-averaged mean square displacement (MSD), which is defined as:

$$M(n) = \lim_{n\to\infty} \frac{1}{N}\sum_{j=1}^{N} d(j,n)^2$$

where:

$$d(j,n) = \sqrt{(p_{j+n} - p_j)^2 + (p_{j+n} - p_j)^2}.$$

is the time lapse of the duration n $(n \ll N)$ starting from the position at time j. It is important to use values of n small enough compared to N, noted n_{cut}, $(n \leq n_{cut})$. A subset of time lags $n_{cut} \in [1, N/10]$ is advised for the computation of each K_c.

As proposed in Gottwald and Melbourne (2009), we compute the modified MSD by substracting the oscillatory term:

$$D(n) = M(n) - E(\phi)^2 \frac{1 - \cos(nc)}{1 - \cos c}.$$

The output of the 0-1 test for chaos is computed by the correlation method as:

$$K_c = corr(\xi, \Delta) \in [-1,1]$$

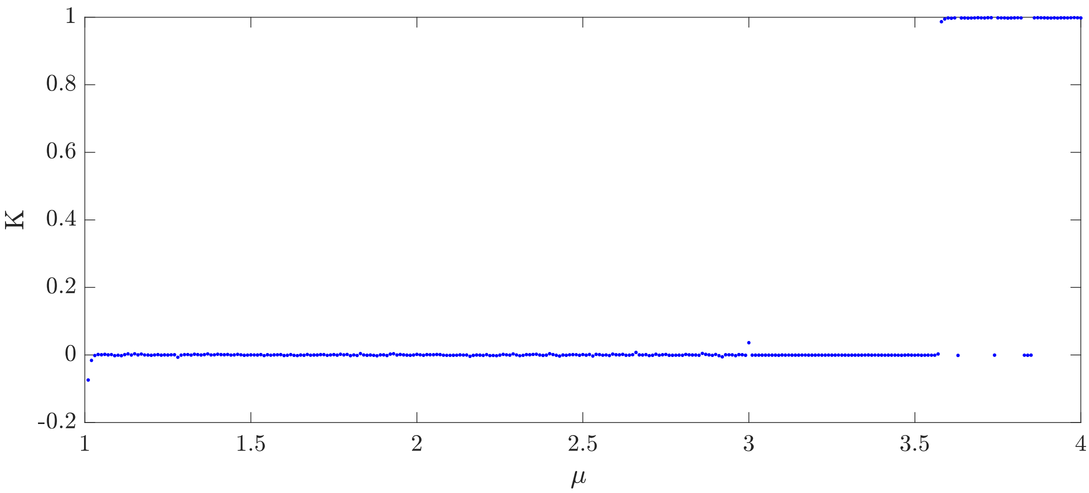
for the vectors $\xi = (1, 2, \ldots, n_{cut})$ and $\Delta = (D_c(1), D_c(2), \ldots, D_c(n_{cut}))$.

After computing K_c for at least a hundred values of c, the final result of the test is:

$$K = median(K_c).$$

The results of the 0-1 test for chaos K for the logistic map (1) is shown in Figure 5. Values of K close to 0 correspond to the periodic behaviour, while values close to 1 correspond to chaos.

Figure 5 Output of the 0-1 test for chaos K versus the parameter μ of the logistic map (1).



IV. Why is Chaos Important and what is mean in economics?

In the following, the focus will be on the definition of the constitutive equation of the Cournot oligopoly model. This equation is then iterated to observe the occurrence of chaos. Finally, the context of chaos theory in the oligopoly model and in economics is then discussed in more detail.

4.1. The Cournot duopoly model

For our purpose we introduce the Cournot model (Puu, 2010). First, we have an isoelastic demand function, where price is denoted p and the supplies of the two competitors are denoted q_1 and q_2. The price is reciprocal to the total supply, so

$$p = \frac{1}{q_1+q_2}. \tag{2}$$

Therefore, the revenues of the companies are

$$pq_1 = \frac{q_1}{q_1+q_2}, \tag{3a}$$

$$pq_2 = \frac{q_2}{q_1+q_2}. \tag{3b}$$

Both competitors produce with constant marginal costs, denoted c_1 and c_2. Their profits, Π_1 and Π_2, can be than obtained as

$$\Pi_1 = \frac{q_1}{q_1+q_2} - c_1 q_1, \tag{4a}$$

$$\Pi_2 = \frac{q_2}{q_1+q_2} - c_2 q_2. \tag{4b}$$

The competitors maximize their profits with respect to q_1 and q_2 respectively. Therefore, we put the partial derivatives equal to zero

$$\frac{\partial \Pi_1}{\partial q_1} = 0, \tag{5a}$$

$$\frac{\partial \Pi_2}{\partial q_2} = 0. \tag{5b}$$

from which we get these reaction functions

$$q_1 = \sqrt{\frac{q_2}{c_1}} - q_2, \tag{6a}$$

$$q_2 = \sqrt{\frac{q_1}{c_2}} - q_1. \tag{6b}$$

So we can rewrite the studied Cournot model as

$$q_1(n+1) = \sqrt{\frac{q_2(n)}{c_1}} - q_2(n), \tag{7a}$$

$$q_2(n+1) = \sqrt{\frac{q_1(n)}{c_2}} - q_1(n). \tag{7b}$$

In order to rule out negative outputs, which is impossible in terms of economy, we rewrite the equations as

$$q_1(n+1) = \begin{cases} \sqrt{\frac{q_{2(n)}}{c_1}} - q_2(n) & c_1 q_2(n) \le 1 \\ 0 & c_1 q_2(n) > 1 \end{cases} \tag{8a}$$

$$q_2(n+1) = \begin{cases} \sqrt{\frac{q_{1(n)}}{c_2}} - q_1(n) & c_2 q_1(n) \le 1 \\ 0 & c_2 q_1(n) > 1 \end{cases} \tag{8b}$$

Now, the tools described in the previous section will be applied to detect dynamics of the Cournot model (8). This model (8) can be generalized for n competitors and the local stability of the model was explored in e.g. (Lampart, 2012) and for Nash equilibria it was proven the following:

Proposition 1 (Lampart, 2012). If $c_i = c$ for any i, then for a dynamical system (X_n, F_n):

(i) $dim\ W^u(f_0) = n\ and\ dim\ W^s(f_0) = 0\ for\ any\ n \geq 2,$

(ii) $dim\ W^u(f_1) = 0\ and\ dim\ W^s(f_1)\ =\ n\ for\ any\ n = 2{,}3,$

(iii) $for\ n = 4\ and\ f_1\ for\ any\ dim\ W^s(f_1) \geq 3,$

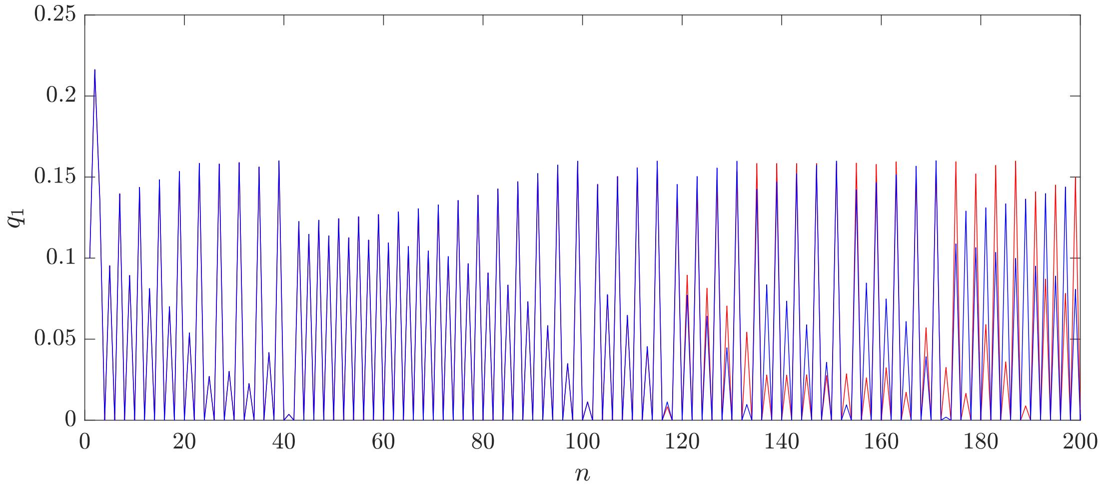
(iv) $dim\ W^u(f_1) = 1\ and\ dim\ W^s(f_1) = n - 1\ for\ any\ n > 4.$

Here f_0 and f_1 stands for system (8) equilibria, f_1 is trivial one, that is zero, and the second one nontrivial; and as usual W^u and W^s stands for stable resp. unstable manifold.

4.2 Sensitive dependence on initial conditions

For showing the system's (8) sensitivity on initial conditions, two initial points close to each other were chosen, $x_{01} = 0.1$ and $x_{02} = 0.1 + 10^{-9}$, in red and blue respectively in Figure 6. We observe that the trajectories start to differ significantly after 120 iterations. The parameters were chosen as $c_1 = 1$ and $c_1 = 6.25$.

Figure 6 Two trajectories of q_1 of the Cournot model (8) obtained by iterating the system from two different initial conditions.



How a chaotic Cournot oligopoly system behaves depends heavily on its initial conditions, since each new position is based on the movement of q_1 from the previous situation. This method of computation says that any error in the measurement of initial conditions immensely affects the next measurement for calculating the point of impact, in our case how the volume of the production q_1 will evolve. Clearly, the dependence on initial conditions also requires us to abandon traditional statistical approaches that take into account the concept of error.

However, the dependence on initial conditions is also what makes accurate long-term predictions impossible. Even if we fully know all the forces at work, we can never know all the initial conditions with sufficient precision to accurately calculate a long-term forecast. Long-term prediction of chaotic systems is impossible, but understanding chaotic systems forces us to re-examine the nature of prediction, reps. why is it necessary to predict something that is completely determined?

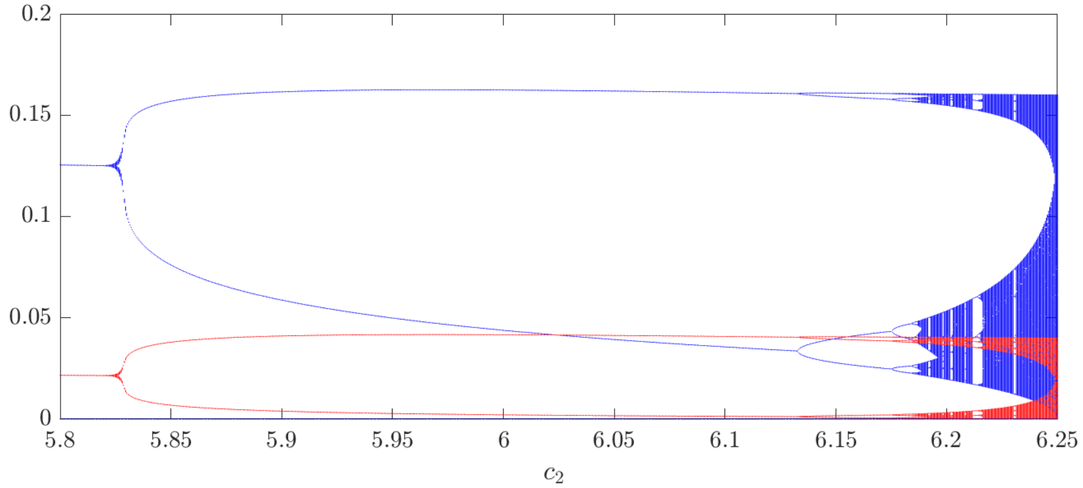
4.3 Bifurcation diagram

The bifurcation diagram shows the states of the system in dependence of the bifurcation parameter, in this case the parameter c_2. The initial conditions were chosen as $q_1(0) = q_2(0) = 0.7$ and $c_1 = 1$. In Figure 7, we see the bifurcation diagram of both variables, q_1 in blue and q_2 in red.

4.4 The Lyapunov exponents

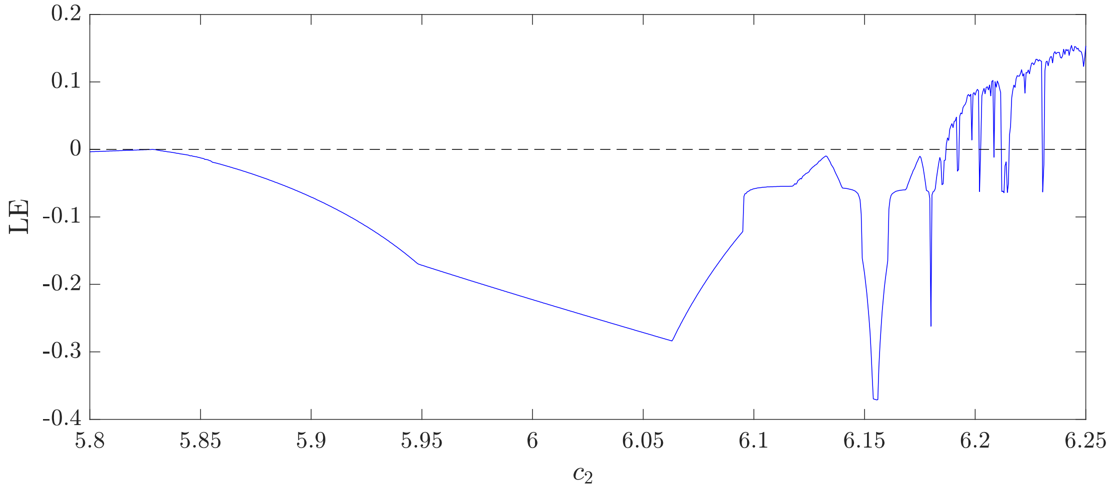
The Lyapunov exponent was computed with initial conditions as $q_1(0) = q_2(0) = 0.7$ and $c_1 = 1$ and $c_2 \in [5.8, 6.25]$. The values were computed using the QR method (Von Bremen et al., 1997).

Figure 7 The bifurcation diagram of q_1 (blue) and q_2 (red) of the Cournot model (8).



The values of the maximal Lyapunov exponent (MLE) for the Cournot model (8) are shown in Figure 8. Negative values means periodic behaviour. Exponent is equal to 0 in the points where bifurcation happens. Chaotic behaviour is characterized by positive values of the MLE towards the end of the studied interval.

Figure 8 The values of the maximal Lyapunov exponent LE versus the parameter c_2 of the Cournot model (8).



4.5 The 0-1 test for chaos

The 0-1 test for chaos was computed with initial conditions $q_1(0) = q_2(0) = 0.7, c_1 = 1$ and $c_2 \in [5.8, 6.25]$.

The difference of a bounded and unbounded trajectory in the (p, q)-plane is shown in Figure 9 for two values of the parameter c_2. The results of the 0-1 test for chaos K for the Cournot model (8) are shown in Figure 10. Values of K close to 0 correspond to the periodic movement, while values close to 1 correspond to chaotic regions. There are also few values of the parameter c_2, for which K is neither close to 0 nor 1. For these parameters, the 0-1 test failed to identify the dynamics of the system.

Figure 9 Plot of translation variables p versus q for c_2 = 5.8 (regular case) and c_2 = 6.25 (chaotic case).

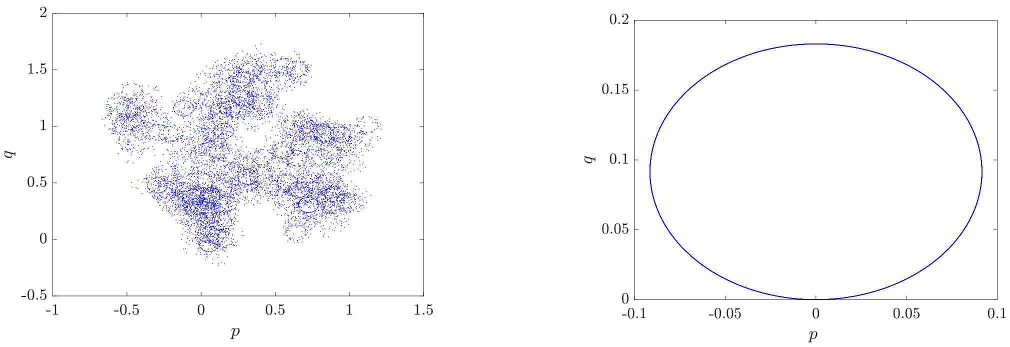
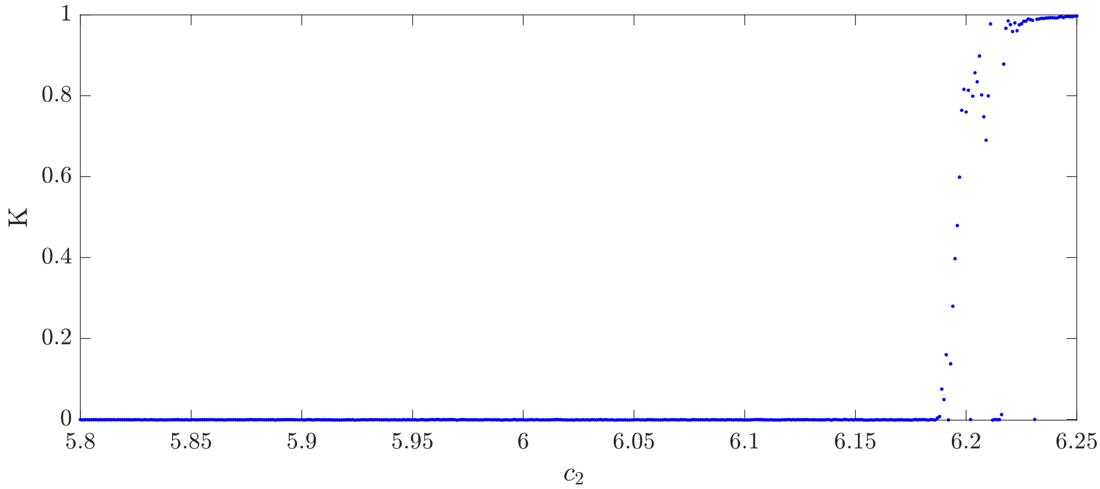


Figure 10 Output of the 0-1 test for chaos K versus the parameter c_2 of the Cournot model (8).



Chaos is not a state, but a continuum of states ranging from complete stability to patterns of events that are incomprehensibly complex. Low order chaos is understandable and predictable. High-order chaos appears random only because we don't understand it. Thus, what is chaotic is not determined by the nature of the events we study, but by the low level of our understanding.

Bifurcation arises when a qualitative change in the behaviour of a dynamical system occurs when some parameter of the dynamical system is changed. In our case, it is when the parameter c_2 is changed that the steady state equilibrium point crosses the hyperbolicity boundary. The periodicity is still predictable in a certain state, however, there may be a negative gain in this case, which is not desirable from the economic point of view. At the level of parameter c_2 = 6.25, unpredictable chaos already occurs, which was confirmed by the 0-1 test for chaos. But what do these facts mean in economics or for the case of Cournot oligopoly? To begin with, let us return to the fact that this system moves in a phase space where it is important to observe changes in the system not its absolute value. If we want to control the volume of sales in an oligopoly structure, we need to control the trajectory reflected by the bifurcation diagram. In business, we are trying to manage the evolving performance patterns of a complex system. To do this, we should use tools that focus on changing the performance trajectories of these systems. The complex pattern shown in Figure 7 can be generated by iterating the appropriate nonlinear equation with specific initial values. Conversely, an empirical approach can be used. One can collect data regarding the performance of a particular system, do simple calculations, and obtain a limit cycle that accurately depicts the system behaviour and closely approximates the attractor. However, in macroeconomics, it is difficult to obtain the necessary amount of data. In our microeconomic case, if we knew the exact equation expressing the attractor of the behaviour of oligopolistic firms then we could accurately calculate all future values of sales volume in each oligopoly game simply by iterating the given equation, or we could predict the future. However, this is not possible since we do not know the given attractors precisely.

It is not disappointing that we will never know those attractors accurately. However, around each attractor there is a region in which its attraction operates, the so-called ”basin of attraction”, a region in which any level of power will be attracted to follow the attractor. This means that we do not need to know the attractor exactly, and yet we can benefit from knowledge of its existence. Another fact to remember is that each firm operates in each institutional environment characterized by a competitive, sectoral, or legal environment. This constitutes the limit cycles of attractors. That is, the environment in which the system moves and is limited. The Cournot oligopoly model is just a simplification of such an environment under certain conditions and gives us a picture of the interaction of two competing firms. From an economic point of view, such a representation is quite realistic. Consider that, unlike perfect competition or monopoly, oligopoly structures are more common in economics. While monopolistic competition reflects the actual economic environment more realistically, it’s very complexity would require knowledge of many attractors.

Why, then, introduce chaos theory in the context of a Cournot oligopoly? The answer is simple and follows from the above. It is precisely the fact of the limited number of players in the market and the specific structural characteristics of the firm (e.g. the energy market) that makes it possible to predict the behaviour of players on the basis of game theory. By appropriately timing decisions affecting the control parameters of the system, we can create more systematic oscillations that reduce risk, increase resource utilisation, and improve our ability to predict firm performance.

V. Conclusion

Chaos theory has caused a change in thinking and view of a complex system such as the economic environment. While theories in the exact sciences based on chaos have had many successes, the situation in economics is much more complex, as the subject of their study is people and their behaviour. Chaos is generated by many economic models, such as the Cournot oligopoly model. This model, which simplifies economic reality, is appropriate because it does not represent a system as complex as monopolistic competition and, thanks to the development of game theory and knowledge of the structural characteristics of the firm, limit cycles of attractors can be appropriately derived. Subsequently, chaos can be controlled and managed. Thus, the stability conditions of the Nash equilibrium can be investigated and stability disturbances can be studied. Of particular importance here is the change and sensitivity to initial conditions that are typical of a chaotic system.

Proving the existence of deterministic chaos from real economic data is much more complicated. Especially in the field of macroeconomics, a large amount of data is needed, which is often missing.

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